Problem Set 4 (Solutions) (Due 2/5/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work. Note: you may not use any results proved in class on 2/2/21 or 2/4/21 to solve these problems (i.e. anything after Section 46 of the textbook cannot be used).

P1. (a) Let $w = \cos^{-1} z$. Show that $\cos^{-1} z = -i \log(z + i(1 - z^2)^{1/2})$ by solving the equation

$$z = \frac{e^{iw} + e^{-iw}}{2}.$$

(b) The expression for $\cos^{-1} z$ is multiple-valued. By taking the principal branch of log and the principle branch of the square root[1] (making $\cos^{-1} z$ single-valued and analytic), show that

$$\frac{d}{dz} \cos^{-1} z = \frac{-1}{(1 - z^2)^{1/2}}.$$

Solution. (a) Multiply through by $2e^{iw}$ to get a quadratic equation:

$$e^{2iw} - 2ze^{iw} + 1 = 0.$$

By the quadratic formula from PSet 1,

$$e^{iw} = \frac{2z + (4z^2 - 4)^{1/2}}{2} = z + (z^2 - 1)^{1/2} = z + i(1 - z^2)^{1/2}.$$

By definition of the logarithm (there is no need to "take the log of both sides" - we solved this kind of equation in full generality in the lecture)

$$iw = \log(z + i(1 - z^2)^{1/2}).$$

Hence, $\cos^{-1} z = w = -i \log(z + i(1 - z^2)^{1/2})$.

(b) This is a straight forward computation using the chain rule.

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P2. Let $z : [a, b] \to \mathbb{C}$ be a parameterization of a smooth arc C and suppose $f(z)$ is analytic at a point $z_0 = z(t_0)$ on C. Show that[2]

$$(f \circ z)'(t) = f'(z(t))z'(t)$$

when $t = t_0$.

[1] Evidently, the value of $\frac{d}{dz} \cos^{-1} z$ depends on the branch of square root we choose.

[2] Hint: Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$. Then $(f \circ z)(t) = u(x(t), y(t)) + iv(x(t), y(t))$. Use the chain rule from vector calculus to compute $(f \circ z)'$ and then use the Cauchy-Riemann equations.

Proof. Write $f(z) = u(x,y) + iv(x,y)$ and $z(t) = x(t) + iy(t)$ so that

$$(f \circ z)(t) = u(x(t), y(t)) + iv(x(t), y(t)).$$

Since f is analytic at z_0, there is a neighborhood of z_0 on which: the derivative exists, the Cauchy-Riemann equations are satisfied, and $f'(z) = u_x(x,y) + iv_x(x,y)$. If $z(t)$ belongs to this neighborhood, then we have

$$\begin{aligned}
(f \circ z)'(t) &= f'(z(t)) \\
&= u_x(x(t), y(t)) + iv_x(x(t), y(t)) \\
&= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + i\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right) && \text{(chain rule)} \\
&= \frac{\partial u}{\partial x}\frac{dx}{dt} - \frac{\partial v}{\partial x}\frac{dy}{dt} + i\left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial x}\frac{dy}{dt}\right) && \text{(Cauchy-Riemann equations)} \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) \\
&= f'(z(t))z'(t).
\end{aligned}$$

Taking $t = t_0$ in the preceding calculation proves the claim. ☕

P3. Let $\alpha, \beta \in \mathbb{R}$. Evaluate the following integral of real-valued functions

$$\int_0^\pi e^{\alpha x} \cos \beta x \, dx \quad \text{and} \quad \int_0^\pi e^{\alpha x} \sin \beta x \, dx$$

simultaneously by computing a *single* integral of a complex-valued function. [3]

Solution. According to Eulers formula, we have

$$\operatorname{Re} \int_0^\pi e^{(\alpha + i\beta)x} \, dx = \int_0^\pi e^{\alpha x} \cos \beta x \, dx \quad \text{and} \quad \operatorname{Im} \int_0^\pi e^{(\alpha + i\beta)x} \, dx = \int_0^\pi e^{\alpha x} \sin \beta x \, dx.$$

Hence, we can compute both integrals simultaneously by computing the single integral

$$\int_0^\pi e^{(\alpha + i\beta)x} \, dx$$

and taking its real and imaginary parts. If $\alpha = \beta = 0$, then the value of the integral is π. Otherwise, at least one of α, β is non-zero. In that case, we have $\frac{d}{dt}\frac{1}{\alpha + i\beta}e^{(\alpha+i\beta)x} = e^{(\alpha+i\beta)x}$. Then by the extension of the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned}
\int_0^\pi e^{(\alpha+i\beta)x} \, dx &= \frac{1}{\alpha + i\beta}\left(e^{(\alpha+i\beta)\pi} - 1\right) \\
&= \frac{\alpha - i\beta}{\alpha^2 + \beta^2}\left(e^{\alpha\pi} \cos \beta\pi + ie^{\alpha\pi} \sin \beta\pi - 1\right) \\
&= \frac{e^{\alpha\pi}(\alpha \cos \beta\pi + \beta \sin \beta\pi) - \alpha}{\alpha^2 + \beta^2} + i\frac{e^{\alpha\pi}(\alpha \sin \beta\pi - \beta \cos \beta\pi) + \beta}{\alpha^2 + \beta^2}.
\end{aligned}$$

Thus,

$$\int_0^\pi e^{\alpha x} \cos \beta x \, dx = \frac{e^{\alpha\pi}(\alpha \cos \beta\pi + \beta \sin \beta\pi) - \alpha}{\alpha^2 + \beta^2}$$

[3]Hint: consider the complex-valued function of a real variable $f(x) = e^{\alpha x} \cos \beta x + ie^{\alpha x} \sin \beta x$.

and
$$\int_0^{\pi} e^{\alpha x}\sin\beta x\,dx = \frac{e^{\alpha\pi}(\alpha\sin\beta\pi - \beta\cos\beta\pi)+\beta}{\alpha^2+\beta^2}.$$

P4. Let $z_1, z_2 \in \mathbb{C}$. Compute the integral
$$\int_C 1\,dz$$
where C is any contour joining z_1 to z_2.

Proof. The proof is essentially the technique used in one of the implications of the Fundamental Theorem of Contour Integrals. First, assume that C is a smooth arc joining z_1 to z_2. Let $z(t) : [a,b] \to \mathbb{C}$ be a parameterization of C. Then $\frac{d}{dz}z(t) = z'(t)$, so by the extension of the Fundamental Theorem of Calculus we can write
$$\begin{aligned}\int_C 1\,dz &= \int_a^b z'(t)\,dt\\ &= z(b) - z(a)\\ &= z_2 - z_1.\end{aligned}$$

Now, assume that C is any contour. Then C is a sum of finitely many smooth arcs, say $C = \sum_{i=1}^n C_i$. Each arc C_i joins a point $w_i \in \mathbb{C}$ to a point $w_{i+1} \in \mathbb{C}$ where $z_1 = w_1$ and $z_2 = w_{n+1}$. We see that
$$\begin{aligned}\int_C 1\,dz &= \int_{\sum_{i=1}^n C_i} 1\,dz\\ &= \sum_{i=1}^n \int_{C_i} 1\,dz && \text{(properties of the integral)}\\ &= \sum_{i=1}^n w_{i+1} - w_i && \text{(C_i is a smooth arc)}\\ &= w_{n+1} - w_1 && \text{(telescoping summation)}\\ &= z_2 - z_1\end{aligned}$$

This completes the solution.

P5. Let C denote the unit circle with positive orientation. Compute the integral
$$\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz$$
for any integers $0 \le k \le n$.[4]

Solution. Parameterize the unit circle as $z(t) = e^{[it}$ with $t \in [0, 2\pi]$. Then $z'(t) = ie^{it}$ and we

[4] Hint: you may assume the binomial theorem holds for complex numbers.

compute:

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1+e^{it})^n}{e^{i(k+1)t}}e^{it}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}(1+e^{it})^n e^{-ikt}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{j=0}^{n}\binom{n}{j}1^{n-j}e^{ijt}\right)e^{-ikt}\,dt \qquad \text{(binomial theorem)} \\
&= \frac{1}{2\pi}\sum_{j=0}^{n}\binom{n}{j}\int_0^{2\pi} e^{ijt}e^{-ikt}\,dt \\
&= \frac{1}{2\pi}\sum_{j=0}^{n}\binom{n}{j}\int_0^{2\pi} e^{i(j-k)t}\,dt
\end{aligned}$$

If $j = k$, then $e^{i(j-k)t} = e^0 = 1$ and we obtain

$$\int_0^{2\pi} e^{i(j-k)t}\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$$

Otherwsie, $j-k \neq 0$ and thus $e^{i(j-k)t}$ has an antiderivative: $\frac{d}{dz}\frac{1}{i(j-k)}e^{i(j-k)t} = e^{i(j-k)t}$. Thus, by the extension of the Fundamental Theorem of Calculus,

$$\int_0^{2\pi} e^{i(j-k)t}\,dt = \frac{1}{i(j-k)}e^{i(j-k)2\pi} - \frac{1}{i(j-k)}e^{i(j-k)0} = 0.$$

Therefore,

$$\frac{1}{2\pi i}\int_C \frac{(1+z)^n}{z^{k+1}}\,dz = \frac{1}{2\pi}\sum_{j=0}^{n}\binom{n}{j}\int_0^{2\pi} e^{i(j-k)t}\,dt = \frac{1}{2\pi}\binom{n}{k}2\pi = \binom{n}{k}.$$

P6. Integrate the function $f(z) = \overline{z}$ over the following contours:

(a) C_1: the line segment joining 0 to $1+i$;

(b) C_2: the line segment joining 0 to 1, following by the line segment joining 1 to $1+i$.

Solution. (a) Parameterize C_1 via $z(t) = t + it$ with $t \in [0, 1]$. Then $z'(t) = 1 + i$ and by definition of the contour integral

$$\begin{aligned}
\int_{C_1}\overline{z}\,dz &= \int_0^1 \overline{t+it}(1+i)\,dt \\
&= \int_0^1 (t-it)(1+i)\,dt \\
&= \int_0^1 2t, dt \\
&= 1.
\end{aligned}$$

(b) The contour C_2 is a sum of two curves, say $C_2 = D_1 + D_2$ where D_1 joins 0 to 1 and D_2 joins 1 to $1+i$. Parameterize D_1 via $z_1(t) = t$ with $t \in [0,1]$ and D_2 via $z_2(t) = 1 + it$ with $t \in [0,1]$. Then by definition of the contour integral

$$\begin{aligned}
\int_{C_2} \overline{z}\, dz &= \int_{D_1} \overline{z}\, dz + \int_{D_2} \overline{z}\, dz \\
&= \int_0^1 \overline{t} \cdot 1\, dt + \int_0^1 \overline{1+it} \cdot i\, dt \\
&= \int_0^1 t\, dt + \int_0^1 i + t\, dt \\
&= \frac{1}{2} + (i + \frac{1}{2}) = 1 + i.
\end{aligned}$$

This completes the solution. Notice that the integrals of $f(z)$ from 0 to $1+i$ are not independent of path. We can conclude that $\overline{z}$ has no antiderivative on any domain containing the curves C_1 and C_2.